The exam's answer

Question No. 1

(16 marks)

(a) Let S={a, b, c, d, e, f} with P(a)=1/16, P(b)=1/16, P(c)=1/8, P(d)=3/16, P(e)=1/4 and P(f)=5/16. Let A={a, c, e}, B={c, d, e, f} and C={b, c, f}. Find:

i) P(A/B).
ii) P(B/C).
iii) $P(C/A^C)$.
iv) $P(A^C/C)$.

Solution

P(A) = P(a) + P(c) + P(e) = 1/16 + 1/8 +1/4 = 7/16 .
P(B) = P(c) + P(d) + P(e) + P(f) = 1/8 + 3/16 +1/4 + 5/16 = 7/8 .
P(C) = P(b) + P(c) + P(f) = 1/16 + 1/8 + 5/16 = 1/2 .

i. $P(A/B) = P(A\cap B)/P(B)$
$A\cap B = \{ (c , e) \}$
$P(A\cap B) = P(c) + P(e) = 1/8 + 1/4 = 3/8$
$P(A/B) = P(A\cap B)/P(B) = 3/8 \div 7/8 = 3/8 * 8/7 = 3/7$.

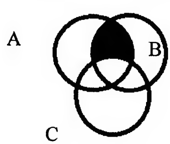
ii. $P(B/C) = P(B\cap C)/P(C)$
$B\cap C = \{ (c , f)\}$
$P(B\cap C) = P(c) + P(f) = 1/8 + 5/16 = 2/16 + 5/16 = 7/16$.
$P(B/C) = P(B\cap C)/P(C) = 7/16 \div 1/2 = 7/16 * 2/1 = 7/8$.

iii. $P(C/A^C) = P(C\cap A^C)/P(A^C)$
$A^C = S - A = \{a , b , c , d , e , f\} - \{a , c , e\} = \{b , d , f\}$.
$C\cap A^C = \{b , f\}$
$P(C\cap A^C) = P(b) + P(f) = 1/16 + 5/16 = 6/16$.
$P(C/A^C) = P(C\cap A^C)/P(A^C) = 6/16 \div (1 - 7/16) = 6/16 \div 9/16 = 2/3$.

iv. $P(A^C/C) = P(A^C\cap C)/P(C) = 6/16 * 2 = 3/4$.

(b) Let A, B, and C be events. **Find an expression, and exhibit the Venn diagram**, for the event that:

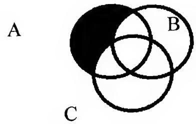
i) A and B, but not C occurs.
Expression is : $(A\cap B) - C = (A-C) \cap (B-C)$



Venn Diagram

ii) Only A occurs.

Expression is : $A-(B \cup C) = (A-B) \cap (A-C)$



Venn Diagram

(c) In a certain college, 25% of the boys and 10% of the girls are studying mathematics. The girls constitute 60% of the students. If a student is selected at random and is studying mathematics, determine the probability that the student is a girl?

Solution

E1={student is a girl} P(E1)= 60/100

E2={student studying math} P(E2)= 16/100

E3={girl studying math} P(E3)= 6/100 = $P(E1 \cap E2)$

Then $P(E1/E2) = P(E1 \cap E2)/P(E2)$

$= (6/100)/(16/100) = 6/16 = 3/8$

Question No. 2

(18 marks)

(a) Find the expectation, variance, and standard deviation of the random variable x with density function P(x) given as:

x	1	3	4	5
P(x)	0.4	0.1	0.2	0.3

Solution

$\mu = E(x) = \sum x\, p(x) = 1*0.4 + 3*0.1 + 4*0.2 + 5*0.3 = 3$

$E(x^2) = \sum x^2 P(x) = 1^2*0.4 + 3^2*0.1 + 4^2*0.2 + 5^2*0.3 = 12$

$\delta^2 = E(x)^2 - \mu^2 = 12^2 - 9 = 3$

$\delta = \sqrt{3} = 1.73$

(b) Prove that for any random variable x:

i) $E(ax+b) = a\,E(x) + b$

ii) $V(ax+b) = a^2\,V(x)$

iii) $E(c) = c$

iv) $V(c) = 0$

where a, b, and c are constants.

Solution

i) E(ax+b)=a E(x)+b

$E(ax+b) = \int_{-\infty}^{\infty}(ax+b)p(x)dx = \int_{-\infty}^{\infty} ax\, p(x)dx + \int_{-\infty}^{\infty} b\, p(x)dx$

$= a\int_{-\infty}^{\infty} xp(x)dx + b\int_{-\infty}^{\infty} p(x)dx = aE(x)+b = R.H.S$

ii) V(ax + b) = a^2 V(x)

V(ax + b)= $E[(ax+b) - E(ax+b)]^2 = E[ax + b - aE(x) + b]^2 =$
$E[ax - aE(x)]^2 = a^2 E[x - \mu]^2 = a^2$ V(x) = R.H.S

iii) E(c) = c

E(x)=$\sum$ x P(x) =$\sum$ c P(c) =$\sum$ c . (1) =c = R.H.S

iv) V(c) = 0

V(x) = $E(x^2) - \mu^2 = c^2 - (E(x))^2 = c^2 - c^2 = 0$ =R.H.S

(c) If the density function f(x) is given by:

$$f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ x-1 & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the distribution function F(x).

Solution

$-\infty \le x \le 0$	,f(x) = 0	F(x) = 0
$0 \le x \le 1$	,f(x) =1-x	F(x) = $F(0)+\int_0^x (1-x)dx = (x-\frac{x^2}{2})$
$1 \le x \le 2$	,f(x) = $x - 1$	F(x) = $F(1) + \int_1^x (x-1)dx = \frac{x^2}{2} - x + 1$
$2 \le x \le \infty$	,f(x) = 0	F(x) = F(2) =2-2+1=1

$$F(x) = \begin{cases} 0 & -\infty \le x \le 0 \\ x - \frac{x^2}{2} & 0 \le x \le 1 \\ \frac{x^2}{2} - x + 1 & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

Question No. 3

(18 marks)

(a) A coin, weighted with P(H)= 3/4 and P(T)= 1/4, is tossed three times. Let x be a random variable denoting the longest string of heads that occurs. Find the distribution, expectation, variance, and standard deviation of x.

Solution

P(H)=3/4 P(T)=1/4

Number of tosses =3

X: denotes the longest string of heads

S={(T,T,T),(T,T,H),(T,H,T),(H,T,T),(H,H,T),(H,T,H),(T,H,H),(H,H,H)}

X(T,T,T)=0 , P(0)=(1/4*1/4*1/4)= 1/64

X(T,T,H)=X(T,H,T)=X(H,T,T)=X(H,T,H)=1 , P(1)=(1/4 * 1/4* 3/4) +(1/4* 3/4* 1/4)+(3/4*1/4*1/4)
=18/64

X(H,H,T)=X(T,H,H)=2 , P(2)=(3/4*3/4*1/4)+1/4*3/4*3/4)=18/64

X(H,H,H)=3 , P(3)=(3/4*3/4*3/4)=27/64

Distribution:

X	0	1	2	3
P(x)	1/64	18/64	18/64	27/64

Expectation:

$\mu = E(x) = \sum x\, P(X) =$ (0)*(1/64) +(1)*(18/64) + (2)*(18/64) + (3)*(27/64) = 2.1

E(x2) = (12)*(18/64) + (22)*(18/64) + (32)*(27/64) = 5.2

Variance:

Vary(x) = $\sigma^2 = E(x^2) - \mu^2 = 5.2 - (2.1)^2 = 0.8$

Standard Deviation Of X :

$\sigma = \sqrt{\sigma^2} == \sqrt{0.8} = 0.9$

(b) Consider the following binomial probability distribution:

$$P(x) = \binom{5}{x} (0.7)^x (0.3)^{5-x} \qquad (x = 0, 1, \ldots, 5)$$

where x is a random variable.

i) How many trials (n) are in the experiment?

ii) What is the value of p, the probability of success?

iii) Graph p(x).

iv) Find the mean and standard deviation of x.

Solution

i) n=5

ii) p=0.7

iii)

$$P(0) = \binom{5}{0} (0.7)^0 (0.3)^5 = 0.00243$$

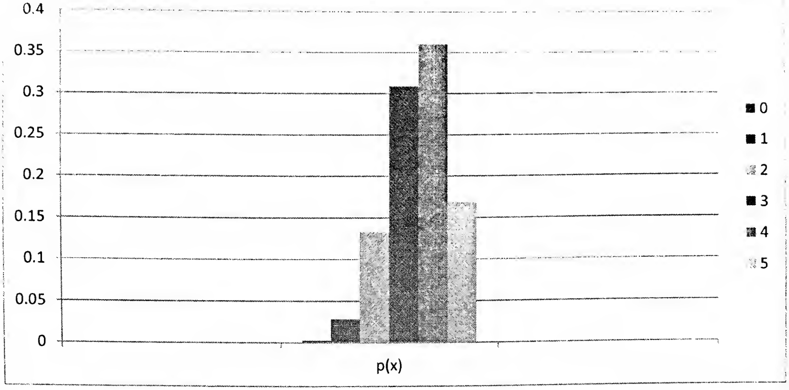
$$P(1) = \binom{5}{1} (0.7) (0.3)^4 = 0.02835$$

$$P(2) = \binom{5}{2} (0.7)^2 (0.3)^3 = 0.1323$$

$$P(3) = \binom{5}{3} (0.7)^3 (0.3)^2 = 0.3087$$

$$P(4) = \binom{5}{4} (0.7)^4 (0.3) = 0.36015$$

$$P(5) = \binom{5}{5} (0.7)^5 (0.3)^0 = 0.16807$$



iv) $E(x)=\sum x\ \ p(x)$

$E(X)=0+(1)*(0.02835)+(2)*(0.1323)+(3)*(0.3087)+(4)*(0.36015)+(5)*(016807)=3.5$

$E(X^2)=\sum X^2\ \ p(x)$

$=0+(1)*(0.02835)+(4)*(0.1323)+(9)*(0.3087)+(16)*(0.36015)+(25)*(016807)=13.3$

$\sigma^2= E(X2\ )-\mu2=13.3-(3.5)^\wedge 2=1.05$

$\sigma=\sqrt{1.05}=1.02$

OR

$\mu=n*p=5*0.7=3.5$

$\sigma^2=n*p*q=5*0.7*0.3=1.05$

$\sigma=\sqrt{1.05}=1.02$

(c) Suppose 2% of items made by a factory are defective. **Find the probability that there are 3 defective items in a sample of 100 items.**

Solution

$$\mathbf{b(3,100,0.02)=\binom{100}{3}(0.02)^3\,(0.98)^{97} \quad = 0.18}$$

<u>Or</u>

$\Lambda = np = 100*0.02 = 2$

$P(k,\lambda) = (\lambda^k\ e^{-\lambda})\ /\ k! = 8\ *\ e^{-2}\ /\ 6 = 0.18$

Question No. 4

(18 marks)

(a) Let x be a random variable with a standard normal distribution Φ. **Find:**

i) $P(x \geq 1.13)$

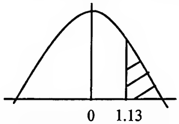
ii) $P(0 \leq x \leq 1.24)$

iii) $P(0.65 \leq x \leq 1.26)$

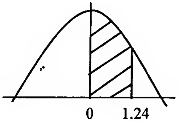
iv) $P(-0.73 \leq x \leq 0)$

Solution

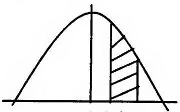
$P(x \geq 1.13)$ is equal to the area under the standard normal curve between 0.5 and 1.13 by using the attached table $P(x \geq 1.13) = 0.5 - 0.3708 = 0.1292$



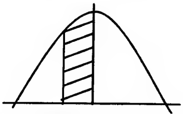
$P(0 \leq x \leq 1.24)$ is equal to the area under the standard normal curve between 0 and 1.24.
$P(0 \leq x \leq 1.24) = 0.3925$



$P(0.65 \leq X \leq 1.26) = P(0 \leq X \leq 1.26) - P(0 \leq X \leq 0.65)$
$= 0.3962 - 0.2422 = 0.1540$



$P(-0.73 \leq x \leq 0) = P(0 \leq x \leq 0.73) = 0.2673$



(b) Let x be a random variable with the standard normal distribution Φ. **Determine the value of t, standard units, if:**

i) $P(0 \leq x \leq t) = 0.4236$

ii) $P(x \leq t) = 0.7967$

iii) $P(t \leq x \leq 2) = 0.1000$

Solution

i) $P(0 \leq x \leq t) = 0.4236$ from the attached tables $t = 1.43$

ii) $P(x \leq t) = 0.7967$

$0.5 + P(0 \leq x \leq t) = 0.7967$

$P(0 \leq x \leq t) = 0.2967$ $t = 0.83$

iii) $P(t \leq x \leq 2) = 0.1000$

$\mathbf{P(0 \leq x \leq 2) - P(0 \leq x \leq t) = 0.1}$

$\mathbf{P(0 \leq x \leq t) = P(0 \leq x \leq 2) - 0.1 = 0.4772 - 0.1 = 0.3772 \qquad t = 1.16}$

(c) A class has 12 boys and 4 girls. If three students are selected at random one after the other from the class, what is the probability that they are all boys?

Solution

P(all boys) = (12/16) * (11/15) * (10/14) = 11/28

Best wishes